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AN HDG METHOD FOR COUPLING MULTISCALE MODELS INVOLVING INTEGRAL BOUNDARY CONDITIONS

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Multiscale coupling is often necessary when modeling complex systems such as ocular mechanics and hemodynamics. In this talk we will address multiscale coupling in terms of spatial dimensions, where a three dimensional model needs to be coupled with a zero dimensional model, leading to the coupling between partial and ordinary differential equations. We enforce the coupling via integral boundary conditions (IBCs). This kind of conditions arises in a variety of different fields. In ocular biomechanics, for example, an IBC arises when a model for the perfusion of the lamina cribrosa in the optic nerve head is coupled with a lumped model for the retrobulbar circulation [1]. Another example is high field magnet (more than 24T) engineering [2], where a target current $\mathrm{I}_{\mathrm{target}} \in \mathbb{R}$ is imposed on a terminal part Γ of the magnet and allows to recover the associated difference of potential between Γ and the other terminal of the magnet. We define $\mathrm{I}_{\mathrm{target}} = \int_\Gamma \underline{j} \cdot \underline{\mathrm{n}}$ where $\underline{j}$ is the current density and n is the outward normal to the terminal part.

We present a novel Hybridized Discontinuous Galerkin (HDG) method for the numerical treatment of problems involving IBCs. The HDG method can be thought of as a mixed DG finite element methos, coded with the use of Lagrange multipliers to weakly enforce the continuity of normal flux on inter-element faces, and then hybridized so that the only variable to be solved for is the collection of Lagrange multipliers [3]. The HDG method has several attractive features: i) it provides optimal approximation of both primal and flux variables, which is a very desirable property for applications where a good approximation of the flux is needed; ii) it requires less globally coupled degrees of freedom than DG methods of comparable accuracy; iii) it allows local element-by-element postprocessing of the primal and flux variables to obtain new approximations with enhanced accuracy and conservation properties. We will show the well-posedness of our novel HDG method and discuss its parallel implementation in Feel++, a C++ library that supports high performance computing up to thousands of cores. Finally, some numerical results in ocular biomechanics and high field magnet engineering will be presented.

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